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Symmetry Reduction From Interactions to Particles

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The hypercharge-isospin-color symmetry of the standard model interaction is drastically reduced to a remaining Abelian electromagnetic **U**(1)-symmetry for the particles. It is shown that such a symmetry reduction is a consequence of the central correlation in the internal group as represented by the standard fields where the hypercharge properties are given by the central isospin-color properties. A maximal diagonalizable symmetry subgroup (Cartan torus) of the interaction group for the particles as eigenvectors has to discard either color (confinement) or isospin. An additional diagonalization for the external spin properties which come centrally correlated with the isospin properties enforces the weak isospin breakdown.

1. INTRODUCTION

The interactions in the standard model (Weinberg, 1967) of elementary particles are invariant under the external transformations with the semidirect Poincaré group $\mathbb{R}^4 \overset{\leftarrow}{\times} \mathbf{SO}_0(1, 3)$ (with respect to half-integer spins written with $\mathbf{SL}(\mathbb{C}^2)$ as the twofold covering group of the Lorentz group $\mathbf{SO}_0(1, 3)$) and under the internal operation group defining hypercharge, isospin, and color properties

$$\text{interaction symmetry:} \quad \underbrace{\mathbb{R}^4 \overset{\leftarrow}{\times} \mathbf{SL}(\mathbb{C}^2)}_{\text{external}} \times \underbrace{\mathbf{U}(1) \times \mathbf{SU}(2) \times \mathbf{SU}(3)}_{\text{internal}}$$

The standard interactions are implemented by the 12 internal gauge fields which come as 4-vectors with respect to the external Lorentz group. I shall show later that slight but important changes should be made in this group with respect to the faithfulness of its representation.

There is a dramatic breakdown[2] from the real $(10 + 12)$-parametric Lie symmetries (Bourbaki, 1989; Fulton and Harris, 1991; Helgason, 1978) for the

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[2] A blowup of symmetries as a consequence of a linearization (tangent space expansion) is proposed in Saller (1998).

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interaction to the symmetries for the particles

$$\text{massive particle symmetry:} \quad \underbrace{\mathbb{R} \times \mathbf{SU}(2)}_{\text{external}} \times \underbrace{\mathbf{U}(1)}_{\text{internal}}$$

$$\text{massless particle symmetry:} \quad \underbrace{\mathbb{R} \times \mathbf{U}(1)}_{\text{external}} \times \underbrace{\mathbf{U}(1)}_{\text{internal}}$$

where Wigner's definition (Wigner, 1939) for free particles as unitary irreducible representations of the Poincaré group is used. In this strict sense, confined quarks cannot be classified as particles because they do not have a mass as eigenvalue for the space–time translations. The Poincaré group representations are induced (Mackey, 1968) by representations of direct product subgroups which have a rotation factor—either **SU**(2) with spin numbers $J = 0, \frac{1}{2}, 1, \ldots$ or axial rotation (polarization) **U**(1) with numbers $M = 0, \pm 1, \pm 2, \ldots$ for massive and massless particles, respectively—and a time translation factor $\mathbb{R}$ represented in a rest system with mass m as eigenvalue for massive particles $m^2 = q^2 > 0$ and in a polarization system with the absolute value of the momentum $|\vec{q}| = q_0$ as eigenvalue for massless particles $m^2 = q^2 = 0$, $|\vec{q}| > 0$.

The word symmetry—in connection with multiplicity—is used in its strict sense, for example, as particles, a proton and a neutron may be called an isospin induced or isospin related doublet, but not an isospin symmetric doublet—with their different masses there is no **SU**(2)-symmetry connecting those two particle states. Or, more obviously, the four particles comprising the weak bosons $\{Z^0, W^\pm\}$ and the photon γ are not isospin symmetric triplet-singlet, there is no symmetry transformation left between them.

The internal symmetry reduction from interaction parametrizing fields to asymptotic free particles has two aspects: Nontrivial color **SU**(3)-representations are confined, and no nontrivial color induced multiplets are seen in the particle regime. Nontrivial isospin induced muliplicities remain visible in the case of the hypercharge–isospin breakdown which is asymptotically reduced to an electromagnetic Abelian **U**(1)-symmetry.

As familiar from the energy eigenstates of quantum mechanics, particles are constructed as eigenvectors with respect to a maximally diagonalizable subgroup, including the time translations, with the corresponding weights collecting the eigenvalues for the operations involved, for example, eigenstates for electromagnetic **U**(1)-operations are characterized by integer charge numbers z, spin **SU**(2)-eigenstates with respect to an **SO**(2) $\cong$ **U**(1)-subgroup (third spin direction) by eigenvalues $|J^3| \leq J$ for a spin $(2J + 1)$-plet etc. The weights of massive particles are given by $(m, J^3, z) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}$ with the components for mass, third spin direction, and integer charge-like number (particle–antiparticle) and the weights of massless particles are given by $(|\vec{q}|, M, z) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}$ with the components for momentum absolute value, polarization, and charge. Therefore, the transition from the large interaction symmetry group to the small particle symmetry group has to discuss the problem of maximal diagonalizable subgroups of the interaction group.

2. CENTRAL CORRELATIONS

An important feature of the operation groups where eigenvectors are looked for is their central correlation structure; this will be explained and exemplified using the old example of the quantum mechanical Kepler potential (hydrogen atom (Fock, 1935)) and the internal standard-model-interaction (Hucks, 1991; O'Raifeartaigh, 1986; Saller, 1992, 1993, 1994, 1998) symmetry.

A direct product of two groups $G_1 \times G_2$ becomes centrally correlated by considering the quotient group defined by the classes with respect to a nontrivial subgroup C in the centers of both factors

$$\frac{G_1 \times G_2}{C}, \{1\} \neq C \subseteq \text{ centr } G_1 \cap \text{ centr } G_2$$

The following Lie groups will be considered

$$n = 1, 2, \ldots: \text{ centr } \mathbf{U}(n) \cong \mathbf{U}(1) \supset \mathbb{I}(n) \cong \text{ centr } \mathbf{SU}(n) = \text{ centr } \mathbf{SL}(\mathbb{C}^n)$$

The center of $\mathbf{SU}(n)$ can be written additively as $\mathbb{Z}$ mod n or, multiplicatively, as the cyclotomic group $\mathbb{I}(n)$

$$\mathbb{Z} \text{ mod } n = \{0, 1, \ldots, n-1\} \cong \mathbb{I}(n) = \left\{ e^{\frac{2\pi i k}{n}} \mid k = 0, \ldots, n-1 \right\}$$

Groups $\mathbf{SU}(n)$ and $\mathbf{SU}(m)$ with n and m relatively prime (no common nontrivial divisor), for example, isospin $\mathbf{SU}(2)$ and color $\mathbf{SU}(3)$, cannot be centrally correlated.

A covering group as Lie algebra exponent gives rise to locally isomorphic groups (i.e., with isomorphic Lie algebras), with respect to discrete center subgroups with the familiar examples:

$$\mathbb{R}/\mathbb{Z} \cong \mathbf{U}(1) \cong \mathbf{U}(1)/\mathbb{I}(n)$$

$$k \text{ divisor of } n\text{: } \mathbf{SU}(n)/\mathbb{I}(k), \text{ e.g. } \begin{cases} \mathbf{SU}(2)/\mathbb{I}(2) \cong \mathbf{SO}(3) \\ \mathbf{SL}(\mathbb{C}^2)/\mathbb{I}(2) \cong \mathbf{SO}_0(1,3) \\ \mathbf{SU}(3)/\mathbb{I}(3) \\ \mathbf{SU}(4)/\mathbb{I}(2), \mathbf{SU}(4)/\mathbb{I}(4), \ldots \end{cases}$$

Obviously, the irreducible representations and the weights of a centrum classified group are subsets of those for the unfactorized group.

2.1. The Eigenstate Squares of the Hydrogen Atom

The perihelion conservation in the orbits as solutions of the Kepler Hamiltonian

$$H = \frac{p^{-2}}{2} - \frac{1}{|\vec{x}|}$$

is described by the Lenz-Runge vector $\mathcal{F}$ which defines a 3-parametric invariance in addition to the position rotation $\mathbf{SO}(3)$ invariance with the angular momenta $\mathcal{L}$, as elements of the rotation Lie algebra[3] $\log \mathbf{SO}(3)$

$$\vec{\mathcal{L}} = \vec{x} \times \vec{p}, \qquad \vec{\mathcal{F}} = \vec{p} \times \vec{\mathcal{L}} - \frac{\vec{x}}{r}$$

$$[\vec{\mathcal{L}}, H] = 0, \qquad [\vec{\mathcal{F}}, H] = 0$$

As shown by Fock, these invariances indicate—not repeating all the subleties found in the literature—an interaction symmetry for the bound states with energy $E < 0$ with the real six-dimensional Lie algebra with basis $\{\vec{\mathcal{L}}_{\pm} = \frac{1}{2}(\vec{\mathcal{L}} \pm \frac{\vec{\mathcal{F}}}{\sqrt{-2H}})\}$

$$\log[\mathbf{SU}(2) \times \mathbf{SU}(2)] \cong \mathbb{R}^6$$

Therewith the bound states are acted upon with representations of the direct product two factor group $\mathbf{SU}(2) \times \mathbf{SU}(2)$ involved whose irreducible representations are characterized by two-integer or half-integer "spin" numbers

$$\textbf{irrep}\,[\mathbf{SU}(2) \times \mathbf{SU}(2)] = \textbf{irrep}\,\mathbf{SU}(2) \times \textbf{irrep}\,\mathbf{SU}(2)$$

$$\textbf{irrep}\,\mathbf{SU}(2) \cong \left\{[J] \mid J = 0, \frac{1}{2}, 1, \ldots\right\}$$

$$\textbf{weights}\,\mathbf{SU}(2) = \left\{M \mid M = 0, \pm\frac{1}{2}, \pm 1, \ldots\right\}$$

However, the experimentally observed energy-degenerated multiplets are all squares, that is, characterized by two equal "spin" numbers for both factors

$$[J; J] = [0; 0], \left[\tfrac{1}{2}; \tfrac{1}{2}\right], [1; 1], \ldots$$

$$\text{with multiplicities } (2J+1)^2 = 1, 4, 9, \ldots$$

$$\textbf{weights}[J; J] = \{(M_1, M_2) \mid |M_{1,2}| \leq J\}$$

$2J + 1$ is the principal quantum number with the energy $E = -\frac{1}{2(2J+1)^2}$. For the nonrelativistic hydrogen theory, the internal two spin directions for the electron

[3] Log G denotes the Lie algebra for the Lie group G.

leading to the observed multiplicities $2(2J+1)^2 = 2, 8, 18, \dots$ have to be taken into account by hand.

The correlation between the two "spin" numbers $[J_1; J_2]$ in Kepler dynamics is a consequence of the orthogonality of angular momentum and perihelion vector

$$\vec{\mathcal{L}}\vec{\mathcal{F}} = 0$$

This orthogonality induces a central correlation: The group maximally faithfully represented on the bound states is not $\mathbf{SU}(2) \times \mathbf{SU}(2)$, but a quotient group which correlates the centers of both factors

$$\text{centr } \mathbf{SU}(2) = \{\pm \mathbf{1}_2\} \cong \mathbb{I}(2)$$

The equivalence group is the "synchronizing" cycle $\mathbb{I}(2)$ in the bicycle $\mathbb{I}(2) \times \mathbb{I}(2)$ (Klein group)

$$\text{centr } [\mathbf{SU}(2) \times \mathbf{SU}(2)] = \{(\pm \mathbf{1}_2, \pm \mathbf{1}_2)\} \supset \{(\mathbf{1}_2, \mathbf{1}_2), (-\mathbf{1}_2, -\mathbf{1}_2)\} \cong \mathbb{I}(2)$$

The irreducible representations of the group with the equivalence classes

$$\frac{\mathbf{SU(2)} \times \mathbf{SU(2)}}{\mathbf{I(2)}} \cong \mathbf{SO}(4), \quad \text{centr } \mathbf{SO}(4) = \{\pm \mathbf{1}_4\} \cong \mathbb{I}(2)$$

are characterized by an integer sum of both "spin" numbers $J_1 + J_2 \in \mathbb{N}$. They come in two types

$$\begin{aligned} &J_1 = J_2 = J\colon [J; J] && \text{with } J = 0, \frac{1}{2}, 1, \dots \\ &J_1 \neq J_2\colon [J_1; j_2] \oplus [J_2; J_1] && \text{with } J_1, J_2 = 0, 1, 2, \dots \end{aligned}$$

with the eigenvalues (weights) in the first case either both integer or both half-integer and, in the second case, both integer

$$\begin{aligned} \textbf{weights } \mathbf{SO}(4)_{J_1 = J_2} &= \{(M_1, M_2) \mid 2M_{1,2} \in \mathbb{Z}, M_1 + M_2 \in \mathbb{Z}\} \\ \textbf{weights } \mathbf{SO}(4)_{J_1 \neq J_2} &= \textbf{weights } \mathbf{SO}(3) \times \textbf{weights } \mathbf{SO}(3) \\ \textbf{weights } \mathbf{SO}(3) &= \{M \mid M \in \mathbb{Z}\} \end{aligned}$$

with the defining representations, faithful and not faithful for $\mathbf{SO}(4)$

$$\begin{aligned} J_1 = J_2\colon \left[\frac{1}{2}, \frac{1}{2}\right]\colon \frac{\mathbf{SU(2)} \times \mathbf{SU(2)}}{\mathbf{I}(2)} &\to \mathbf{SO}(4) \\ (-\mathbf{1}_2, \mathbf{1}_2) &\mapsto -\mathbf{1}_4 \\ J_1 \neq J_2\colon [1; 0] \cong [0; 1]\colon \frac{\mathbf{SU(2)} \times \mathbf{SU(2)}}{\mathbf{I}(2)} &\to \mathbf{SO}(3) \cong \mathbf{SU}(2)/\mathbb{I}(2) \\ (-\mathbf{1}_2, \mathbf{1}_2) &\mapsto +\mathbf{1}_3 \end{aligned}$$

This orthogonality enforces even $J_1 = J_2$, that is, it allows only the complex irreducible representations where the weights occupy squares.

2.2. The Hypercharge Correlation with Isospin–Color

The fields of the standard model transform under isospin $\mathbf{SU}(2)$ with the irreducible representations

$$\begin{aligned}&\textbf{irrep } \mathbf{SU}(2) = \left\{[2T] \mid T = 0, \tfrac{1}{2}, 1, \ldots\right\}\\ &\text{multiplicities: } 2T + 1\end{aligned}$$

as well as under color $\mathbf{SU}(3)$ with the irreducible representations characterized by two integers

$$\begin{aligned}&\textbf{irrep } \mathbf{SU}(3) = \left\{[2C_1, 2C_2] \mid C_{1,2} = 0, \tfrac{1}{2}, 1, \ldots\right\}\\ &\text{multiplicities: } (2C_1 + 1)(2C_2 + 1)(C_1 + C_2 + 1)\end{aligned}$$

From now on, I use integers, odd and even, for the weights and representations replacing the half-integers and integers that were used for familiarity in the Kepler dynamics. The integers are the winding numbers $z \in \mathbb{Z}$ characterizing the representations of $\mathbf{U}(1)$-subgroups.

The left-handed quark and antiquark isodoublet color triplet fields are examples of the complex six-dimensional defining dual representations of isospin–color

$$\begin{aligned}&\textbf{irrep } \mathbf{SU}(2) \times \textbf{irrep } \mathbf{SU}(3) = \{[2T; 2C_1, 2C_2]\}\\ &\text{defining representations: } u = [1; 1, 0], \qquad \check{u} = [1; 0, 1]\end{aligned}$$

The totally antisymmetric tensor-powers of the defining representations generate—up to isomorphism—all fundamental representations for isospin and color by the products

$$\bigwedge^n u \otimes \bigwedge^m \check{u}, \quad n, m \in \mathbb{N} \quad \Rightarrow \quad \begin{cases} \text{for } \mathbf{SU}(2)\text{:} & \bigwedge^3 u \cong [1] \cong \bigwedge^3 \check{u}, \quad \text{doublet} \\ \text{for } \mathbf{SU}(3)\text{:} & \begin{cases} \bigwedge^2 u \cong [0, 1], & \text{antitriplet} \\ \bigwedge^2 \check{u} \cong [1, 0], & \text{triplet} \end{cases} \end{cases}$$

Therewith the hypercharge number y of the interaction fields in the standard model is a consequence of their isospin–color powers

$$6y = n - m$$

as shown in the following table:

Field	Symbol	(n, m)	$\mathbf{U}(1)$ $y = \frac{n-m}{6}$	$\mathbf{SU}(2)$ $[2T]$	$\mathbf{SU}(3)$ $[2C_1, 2C_2]$
Left lepton	**1**	(0,3)	$-\frac{1}{2}$	[1]	[0, 0]
Right lepton	**e**	(0, 6)	-1	[0]	[0, 0]
Left quark	**q**	(1, 0)	$\frac{1}{6}$	[1]	[1, 0]
Right up quark	**u**	(4, 0)	$\frac{2}{3}$	[0]	[1, 0]
Right down quark	**d**	(0, 2)	$-\frac{1}{3}$	[0]	[1, 0]
Higgs	**Φ**	(3, 0)	$\frac{1}{2}$	[1]	[0, 0]
Hypercharge gauge	**A**	(0, 0)	0	[0]	[0, 0]
Isospin gauge	**B**	(1, 1)	0	[2]	[0, 0]
Color gauge	**G**	(1, 1)	0	[0]	[1, 1]

The hypercharge is related to the two-ality of the **SU**(2)-representations and the triality (Baird and Biedenharn, 1964) of the **SU**(3)-representations by the modulo relations

$$\text{isospin duality: } 2T \bmod 2 = 6y \bmod 2$$

$$\text{color triality: } 2(C_1 - C_2) \bmod 3 = 6y \bmod 3$$

The centrality (n-ality) k mod n of an **SU**(n)-representations describes the centrum representation involved

$$\mathbb{I}(n) \ni e^{\frac{2\pi i}{n}} \mapsto e^{\frac{2\pi i k}{n}} \in \mathbb{I}(n)$$

for example, faithful for **SU**(2)-representations with duality $2T \bmod 2 = 1$ and for **SU**(3)-representations with triality $2(C_1 - C_2) \bmod 3 = \pm 1$.

This central correlation shows that the group maximally faithfully represented by the fields in the standard model is given by the following classes of the direct product group

$$\frac{\mathbf{U}(1) \times \mathbf{SU}(2) \times \mathbf{SU}(3)}{\mathbf{I}(6)} = \frac{\mathbf{U}(2) \times \mathbf{SU}(3)}{\mathbf{I}(3)} = \frac{\mathbf{SU}(2) \times \mathbf{U}(3)}{\mathbf{I}(2)} = \mathbf{U}(2 \times 3)$$

The representation of the subgroup "synchronizing" both centrums $\mathbb{I}(2) \times \mathbb{I}(3) = \mathbb{I}(6) \subset \mathbf{U}(1) \cap [\mathbf{SU}(2) \times \mathbf{SU}(3)]$

$$\mathbb{I}(6) \times \mathbb{I}(6) = \left\{ \left(e^{\frac{2\pi i k_1}{6}}, e^{\frac{2\pi i k_2}{6}} \right) \mid k_{1,2} = 0, \ldots, 5 \right\}$$
$$\supset \left\{ \left(e^{\frac{2\pi i k}{6}}, e^{\frac{2\pi i k}{6}} \right) \mid k = 0, \ldots, 5 \right\} \cong \mathbb{I}(6)$$

determines the hypercharge numbers as integer multiples of $\frac{1}{6}$.

The eigenvalue spectrum for the representations of the centrally correlated internal group is

$$\textbf{weights}\ \mathbf{U}(2\times 3) = \Big\{[y\|2t; 2c_1, 2c_2] \mid 2t, 2c_{1,2} \in \mathbb{Z}, \\ \text{with } y \in t - \frac{c_1 - c_2}{3} + \mathbb{Z}\Big\}$$

3. CARTAN TORI

A Lie algebra has Cartan subalgebras, for semisimple Lie algebras given by maximal Abelian subalgebras, diagonalizable in a representation. Going from a Lie algebra to its exponent, a Lie group, a Cartan subalgebra gives rise to a Cartan subgroup. A maximal Abelian direct product subgroup of a compact group

$$\mathbf{U}(1)^n = \underbrace{\mathbf{U}(1)\times\cdots\times\mathbf{U}(1)}_{n\ \text{times}}$$

will be called an *n-dimensional Cartan torus* which may be parametrized for each direct factor (“circle”) by

$$\mathbf{U}(1) = \{e^{i\alpha} \mid \alpha \in [0, 2\pi]\}$$

If the dimension of a Cartan torus coincides with the rank of the Lie algebra, the Cartan torus is called *complete* for the group. In general a complete Cartan torus requires a special (orthogonal) basis. There are situations where a Cartan subalgebra does exist, but complete Cartan torus do not.

3.1. A Complete Cartan Torus for SU(n)

The Lie algebra log $\mathbf{SU}(n) \cong \mathbb{R}^{n^2-1}$, $n \geq 2$, in the defining complex n-dimensional representation has a basis consisting of traceless and hermitian generalized Pauli matrices

$$\{\sigma(n)^a\}_{a=1}^{n^2-1}, \qquad \operatorname{tr}\sigma(n)^a = 0, \qquad \sigma(n)^a = (\sigma(n)^a)^\star$$

constructed inductively from the proper Pauli matrices $\sigma(2) = \sigma$. The start for $n \geq 3$ is the embedded Lie subalgebra of $\mathbf{SU}(n-1)$ with

$$\sigma(n+1)^a = \left(\begin{array}{c|c} \sigma(n)^a & 0 \\ \hline 0 & 0 \end{array}\right), \quad a = 1, \ldots, n^2 - 1$$

The new off-diagonal matrices for $a = n^2, \ldots, (n+1)^2 - 2$ come in $(n-1)$ pairs with unit column vectors e and their transposed e^T as illustrated in the first step for the eight Gell–Mann matrices $\sigma(3) = \lambda$

$$\sigma(n+1)^a = \left(\begin{array}{c|c} 0_n & e \\ \hline e^T & 0 \end{array}\right), \qquad \sigma(n+1)^{a+1} = \left(\begin{array}{c|c} 0_n & -ie \\ \hline ie^T & 0 \end{array}\right)$$

$$\sigma(3)^4 = \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 0 \end{array}\right), \qquad \sigma(3)^5 = \left(\begin{array}{cc|c} 0 & 0 & -i \\ 0 & 0 & 0 \\ \hline i & 0 & 0 \end{array}\right)$$

$$\sigma(3)^6 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 1 & 0 \end{array}\right), \qquad \sigma(3)^7 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & -i \\ \hline 0 & i & 0 \end{array}\right)$$

The new diagonal matrix is defined by

$$\sigma(n)^{n^2-1} = \frac{1}{\sqrt{\binom{n}{2}}} \left(\begin{array}{c|c} \mathbf{1}_{n-1} & 0 \\ \hline 0 & -(n-1) \end{array}\right)$$

Therewith the normalization is as for the proper Pauli matrices

$$\operatorname{tr} \sigma(n)^a \sigma(n)^b = 2\delta^{ab}$$

A Cartan subalgebra log $\mathbf{U}(1)^{n-1}$ is spanned by the diagonal matrices

$$\text{Cartan subalgebra basis: } \left\{ i\sigma(n)^{m^2-1} \mid m = 2, 3, \dots, n \right\}$$

whose exponent gives a complete Cartan torus of dimension $n - 1$ (rank of log $\mathbf{SU}(n)$).

The characteristic diagonal element with a nontrivial determinant generates the centrum of $\mathbf{SU}(n)$ and is renormalized to display integer $\mathbf{U}(1)$-winding numbers in the diagonal

$$\mathbf{w}_n = \sqrt{\binom{n}{2}}\sigma(n)^{n^2-1} = \left(\begin{array}{c|c} \mathbf{1}_{n-1} & 0 \\ \hline 0 & -(n-1) \end{array}\right), \qquad \det \mathbf{w}_n = -(n-1)$$

$$\mathbf{U}(1)_{n^2-1} = \{e^{i\alpha \mathbf{w}_n} \mid \alpha \in [0, 2\pi]\}$$

$$e^{\frac{2\pi i}{n}\mathbf{w}_n} = e^{\frac{2\pi i}{n}} \mathbf{1}_n \in \mathbf{U}(\mathbf{1}_n) \cap \mathbf{U}(1)_{n^2-1} = \mathbf{U}(\mathbf{1}_n) \cap \mathbf{SU}(n) \cong \mathbb{I}(n)$$

Here $\mathbf{U}(\mathbf{1}_n) = \mathbf{U}(1)\mathbf{1}_n$ denotes the scalar phase group.

3.2. A Complete Cartan Torus for Hyperisospin

Hypercharge and isospin symmetry with central correlation, called hyperisospin

$$\frac{\mathbf{U}(1) \times \mathbf{SU}(2)}{\mathbf{I}(2)} \cong \mathbf{U}(2)$$

has a Cartan subalgebra in the defining complex two-dimensional representation

$$\left\{i\alpha_0\mathbf{1}_2 + i\alpha_3\sigma^3 \mid \alpha_{0,3} \in [0, 2\pi]\right\} \cong \mathbb{R}^2$$

Its exponent has the scalar hypercharge and the third isospin component phase group as factors which, however, are no direct factors for a torus

$$e^{i\alpha_0\mathbf{1}_2 + i\alpha_3\sigma^3} \in \mathbf{U}(\mathbf{1}_2) \circ \mathbf{U}(1)_3$$

The parametrization has inherited the following ambiguity from the common centrum $\mathbf{U}(\mathbf{1}_2) \cap \mathbf{SU}(2) \cong \mathbb{I}(2)$

$$(\alpha_0, \alpha_3) = (\pi, 0) \cong (0, \pi), \qquad e^{i\pi\mathbf{1}_2} = e^{i\pi\sigma^3} = -\mathbf{1}_2 \in \mathbb{I}(2) \cong \mathbf{U}(\mathbf{1}_2) \cap \mathbf{SU}(2)$$

A Cartan torus of $\mathbf{U}(2)$ arises with a projector basis containing two orthogonal elements

$$e^{i\alpha_+ \frac{\mathbf{1}_2 + \sigma^3}{2}} e^{i\alpha_- \frac{\mathbf{1}_2 - \sigma^3}{2}} \in \mathbf{U}(1)_+ \times \mathbf{U}(1)_-, \qquad \alpha_\pm = \alpha_0 \pm \alpha_3$$

$$\mathcal{P}_\pm(2) = \frac{\mathbf{1}_2 \pm \sigma^3}{2}, \qquad \mathcal{P}_+(2)\mathcal{P}_-(2) = 0$$

For the general case,

$$\frac{\mathbf{U}(1) \times \mathbf{SU}(n)}{\mathbf{I}(n)} \cong \mathbf{U}(n), \qquad \mathbf{U}(\mathbf{1}_n) \cap \mathbf{SU}(n) \cong \mathbb{I}(n)$$

the exponent of a Cartan subalgebra in the defining complex n-dimensional representation

$$\left\{ i\alpha_0\mathbf{1}_n + i\sum_{m=2}^{n} \alpha_{m^2-1}\sigma(n)^{m^2-1} \,\middle|\, \alpha_{0,m} \in [0, 2\pi] \right\} \cong \mathbb{R}^n$$

gives an Abelian group where the scalar phase factor is correlated with the centrum generating factor

$$\mathbf{U}(\mathbf{1}_n) \circ \mathbf{U}(1)_{n^2-1} \times \mathbf{U}(1)^{n-2}, \qquad e^{i\alpha\mathbf{w}_n} \in \mathbf{U}(1)_{n^2-1}$$

$$\text{e.g., for } \mathbf{U}(3)\text{:} \quad \mathbf{U}(\mathbf{1}_3) \circ \mathbf{U}(1)_8 \times \mathbf{U}(1)_3, \qquad e^{i\alpha\mathbf{w}_3} \in \mathbf{U}(1)_8, \quad \mathbf{w}_3 = \sqrt{3}\lambda^8$$

A Cartan torus comes with the appropriate projectors $\mathcal{P}_\pm(n)$ and parameters $\alpha_\pm$

$$\mathbf{U}(1)_+ \times \mathbf{U}(1)_- \times \mathbf{U}(1)^{n-2}, \qquad e^{i\alpha_+\mathcal{P}_+(n)} e^{i\alpha_-\mathcal{P}_+(n)} \in \mathbf{U}(1)_+ \times \mathbf{U}(1)_-$$

$$\text{with} \begin{cases} \mathcal{P}_+(n) = \dfrac{(n-1)\mathbf{1}_n + \mathbf{w}_n}{n}, & \alpha_+ = \alpha_0 + \dfrac{\alpha_{n^2-1}}{\sqrt{\binom{n}{2}}} \\[2ex] \mathcal{P}_-(n) = \dfrac{\mathbf{1}_n - \mathbf{w}_n}{n}, & \alpha_- = \alpha_0 - (n-1)\dfrac{\alpha_{n^2-1}}{\sqrt{\binom{n}{2}}} \end{cases}$$

$$\mathcal{P}_+(n)\mathcal{P}_-(n) = 0, \qquad (\mathbf{w}_n)^2 = (n-1)\mathbf{1}_n - (n-2)\mathbf{w}_n$$

For the groups $\mathbf{U}(n)$ with rank n Lie algebras there exist complete Cartan tori.

3.3. A Complete Cartan Torus for the Hydrogen Atom

For the nonrelativistic hydrogen bound states, an exponentiated Cartan subalgebra of log[**SU**(2) × **SU**(2)] with basis $\{i\vec{\sigma} \otimes \mathbf{1}_2, \mathbf{1}_2 \otimes i\vec{\tau}\}$ in the defining quartet representation

$$\text{Cartan algebra } \left\{i\alpha_3\sigma^3 \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes i\beta_3\tau^3\right\} \cong \mathbb{R}^2$$

$$e^{i\alpha_3\sigma^3} \otimes e^{i\beta_3\tau^3} = \begin{pmatrix} e^{i(\alpha_3+\beta_3)} & 0 & 0 & 0 \\ 0 & e^{i(\alpha_3-\beta_3)} & 0 & 0 \\ 0 & 0 & e^{-i(\alpha_3-\beta_3)} & 0 \\ 0 & 0 & 0 & e^{-i(\alpha_3+\beta_3)} \end{pmatrix} \in \mathbf{U}(1)_3 \circ \mathbf{U}(1)_3$$

$$\text{parameters: } \{\alpha_3 + \beta_3, \alpha_3 - \beta_3\}$$

leads to a complete Cartan torus via a basis of orthogonal generators $\mathcal{L}_\pm$ for coordinates $\gamma_\pm$

$$e^{i\alpha_3\sigma^3} \otimes e^{i\beta_3\tau^3} = e^{i\gamma_+\mathcal{L}_+^3} e^{i\gamma_-\mathcal{L}_-^3} \in \mathbf{U}(1)_+ \times \mathbf{U}(1)_-$$

$$\mathcal{L}_\pm^3 = \frac{\sigma^3 \otimes \mathbf{1}_2 \pm \mathbf{1}_2 \otimes \tau^3}{2}, \qquad \mathcal{L}_+^3\mathcal{L}_-^3 = 0, \qquad \gamma_\pm = \alpha_3 \pm \beta_3$$

$\mathcal{L}_+^3 = \mathcal{L}^3$ is the third component of the angular momenta log **SO**(3), $\mathcal{L}_-^3 \sim \mathcal{F}^3$ is proportional to the third component of the perihelion vector.

In the case of two special groups, centrally correlatable for dimensions with a common nontrivial factor

$$\frac{\mathbf{SU}(n) \times \mathbf{SU}(m)}{\mathbf{I}(k)}, \qquad \mathbb{I}(k) \subset \mathbb{I}(n) \cap \mathbb{I}(m), \qquad n, m, k \geq 2$$

the exponent of a Cartan Lie subalgebra is centrally correlated by the **U**(1)'s generated by $\mathbf{w}_n$ and $\mathbf{w}_m$:

$$\mathbf{U}(1)_{n^2-1} \circ \mathbf{U}(1)_{m^2-1} \times \mathbf{U}(1)^{n+m-4}$$

$$\mathbf{U}(1)_{n^2-1} \circ \mathbf{U}(1)_{m^2-1}\text{-Lie algebra}: \{i\alpha\mathbf{w}_n \otimes \mathbf{1}_m + \mathbf{1}_n \otimes i\beta\mathbf{w}_m\} \cong \mathbb{R}^2$$

$$e^{i\alpha\mathbf{w}_n} \otimes e^{i\beta\mathbf{w}_m} \in \mathbf{U}(1)_{n^2-1} \circ \mathbf{U}(1)_{m^2-1}$$

In general, there arise four parameters

$$e^{i\alpha\mathbf{w}_n} \otimes e^{i\beta\mathbf{w}_m} \cong \begin{pmatrix} e^{i[\alpha+\beta]} & 0 & 0 & 0 \\ 0 & e^{i[\alpha-(m-1)\beta]} & 0 & 0 \\ 0 & 0 & e^{-i[(n-1)\alpha-\beta]} & 0 \\ 0 & 0 & 0 & e^{-i[(n-1)\alpha+(m-1)\beta]} \end{pmatrix}$$

$$\text{parameters: } \{\alpha + \beta, \alpha - (m-1)\beta, (n-1)\alpha - \beta, (n-1)\alpha - (m-1)\beta\}$$

which, only for the hydrogen symmetry with $n = m = 2$, allows an orthogonal Cartan subalgebra basis leading to a complete Cartan torus.

3.4. No Complete Cartan Torus for Hypercharge–Isospin–Color

The internal interaction symmetry $\mathbf{U}(2 \times 3) = \frac{\mathbf{U}(1)\times\mathbf{SU}(2)\times\mathbf{SU}(3)}{\mathbf{I}(2)\times\mathbf{I}(3)}$ has a defining complex six-dimensional representation for its Lie algebra with rank 4

$$\log[\mathbf{U}(1) \times \mathbf{SU}(2) \times \mathbf{SU}(3)] = \{i\alpha_0 \mathbf{1}_2 \otimes \mathbf{1}_3 + i\vec{\alpha}\vec{\tau} \otimes \mathbf{1}_3 + \mathbf{1}_2 \otimes i\vec{\beta}\vec{\lambda}\} \cong \mathbb{R}^{12}$$

$$\text{Cartan subalgebra: } \{i\alpha_0 \mathbf{1}_2 \otimes \mathbf{1}_3 + i\alpha_3\tau^3 \otimes \mathbf{1}_3 + \mathbf{1}_2 \otimes i(\beta_3\lambda^3 + \beta_8\lambda^8)\} \cong \mathbb{R}^4$$

using three Pauli matrices $\vec{\tau}$ (isospin) and eight Gell-Mann matrices $\vec{\lambda}$ (color).

The exponentiated Lie algebra has three correlated factors generated with $\mathbf{w}_2 = \sigma^3$ and $\mathbf{w}_3 = \sqrt{3}\lambda^8$

$$e^{i\alpha_0\mathbf{1}_2\otimes\mathbf{1}_3 + i\alpha_3\tau^3\otimes\mathbf{1}_3 + \mathbf{1}_2\otimes i\beta_8\lambda^8} \in \frac{\mathbf{U}(\mathbf{1}_6) \circ \mathbf{U}(\mathbf{1})_3 \circ \mathbf{U}(\mathbf{1})_8 \times \mathbf{U}(\mathbf{1})}{\mathbf{U}(\mathbf{1}_6) \circ \mathbf{U}(\mathbf{1})_3 \circ \mathbf{U}(\mathbf{1})_8}$$

The relevant parameter combinations in the four phases that arise

$$\text{parameters}: \left\{(\alpha_0 \pm \alpha_3) + \frac{\beta_8}{\sqrt{3}}, (\alpha_0 \pm \alpha_3) - \frac{2\beta_8}{\sqrt{3}}\right\}$$

cannot be disentangled with an orthogonal basis for a representation of the direct product $\mathbf{U}(1) \times \mathbf{U}(1) \times \mathbf{U}(1)$.

There exists a complete Cartan torus $\mathbf{U}(1)_+ \times \mathbf{U}(1)_-$ for hyperisospin $\mathbf{U}(2)$, parametrized with $\{\alpha_0 \pm \alpha_3\}$, and a complete Cartan torus $\mathbf{U}(1)_+ \times \mathbf{U}(1)_- \times \mathbf{U}(1)_3$ for hypercolor $\mathbf{U}(3)$, parametrized with $\{\alpha_0 + \frac{\beta_8}{\sqrt{3}}, \alpha_0 - \frac{2\beta_8}{\sqrt{3}}, \beta_3\}$. However, a complete Cartan torus $\mathbf{U}(1)^4$ for faithful representations of the internal $\mathbf{U}(2 \times 3)$-interaction symmetry does not exist.

4. EIGENVECTOR BASES FOR CORRELATED GROUPS

A semisimple Lie algebra, and also log $\mathbf{U}(n)$, allows—for any finite-dimensional representation vector space—a basis of eigenvectors for a Cartan subalgebra. A Lie algebra representation also involving nontrivial nilpotent transformations need not have an eigenvector basis (_______; Boerner, 1955).

Eigenvectors of a Cartan subalgebra need not remain eigenvectors for the exponentiated Cartan algebra. However, eigenvectors of a direct product of Abelian groups—of a Cartan torus in the compact case—are needed in the definition of particles (eigenstates).

In the following, "eigenvectors of a Lie algebra" and "eigenvectors of a Lie group" are the acronyms for "eigenvectors of a Cartan subalgebra" and "eigenvectors of a maximal direct product Abelian subgroup," in the case of a compact Lie group of a maximal Cartan torus. With the choice of an eigenvector basis (or of a Cartan subalgebra or of a Cartan torus) the original full symmetry seems to be broken. However, the full symmetry remains in the set with all possible eigenbases, for example, for spin $\mathbf{SU}(2)$ with a complete Cartan torus $\mathbf{U}(1)_3$: The third direction choice to measure spin eigenvalues can be replaced equivalently by any direction.

Since a correlation of two Lie groups $G_1 \times G_2$ via a discrete centrum C does not change the Lie algebra

$$\log \frac{G_1 \times G_2}{C} = \log[G_1 \times G_2] = \log G_1 \oplus \log G_2$$

a case can arise where there exists an eigenvector basis for the Lie algebra representation space but not for the correlated group. This is the case for compact groups without a complete Cartan torus, especially for the internal interaction symmetry group.

4.1. An Eigenvector Basis for U(n)

If a represented compact group has a complete Cartan torus there exists an eigenvector basis of the representation vector space—exemplified for $\mathbf{U}(n)$ and obviously true also for $\mathbf{SU}(n)$.

The diagonals of the log $\mathbf{U}(n)$-Cartan subalgebra basis in the defining representation

$$\left\{i\mathbf{1}_n, i\sigma(n)^{m^2-1} \mid m = 2, 3, \ldots, n\right\}$$

taken as columns in the following $(n \times n)$-scheme (between $\| \cdots \|$) display—up to i—the eigenvalues as components of the weights (in the lines) with the eigenvectors

$$e^1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e^n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$
\begin{array}{c|c|cccccc|}
e^1: & 1 & 1 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{\binom{n}{2}}} \\
e^2: & 1 & -1 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{\binom{n}{2}}} \\
e^3: & 1 & 0 & -\frac{2}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{\binom{n}{2}}} \\
e^4: & 1 & 0 & 0 & -\frac{3}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{\binom{n}{2}}} \\
\vdots & \vdots & & & & \vdots & \\
e^n: & 1 & 0 & 0 & 0 & \cdots & -\frac{n-1}{\sqrt{\binom{n}{2}}}
\end{array}
$$

For example, for $\log \mathbf{U}(1)$ in the left upper (1×1)-matrix, for $\log \mathbf{U}(2)$ in the left upper (2×2)-matrix, etc. A geometrical aside, not really surprising with the permutation group as symmetry group for the fundamental $\mathbf{SU}(n)$-weights: Erasing the first column with the $1'$s for $\log \mathbf{U}(1)$, the remaining n lines ($\mathbf{SU}(n)$-weights with $n-1$ components between $| \ldots \|$) give the corners of a regular fundamental simplex (distance, triangle, tetraeder, etc.) centered at the origin of $\mathbb{R}^{n-1}$.

The Lie algebra for the correlated group $\mathbf{U}(\mathbf{1}_n) \circ \mathbf{U}(1)_{n^2-1}$ has 2-component weights,

$$
\text{for basis } \{i\mathbf{1}_n, i\mathbf{w}_n\}: \quad
\begin{array}{c|c|c|}
e^1: & 1 & 1 \\
e^2: & 1 & 1 \\
\vdots & \vdots & \vdots \\
e^{n-1}: & 1 & 1 \\
e^n: & 1 & -(n-1)
\end{array}
$$

$$
\text{for projector basis } \{i\mathcal{P} \pm (n)\}: \quad
\begin{array}{c|c|c|}
e^1: & 1 & 0 \\
e^2: & 1 & 0 \\
\vdots & \vdots & \vdots \\
e^{n-1}: & 1 & 0 \\
e^n: & 0 & 1
\end{array}
$$

Obviously, the eigenvectors keep their property for the complete Cartan torus $\mathbf{U}(1)_+ \times \mathbf{U}(1)_- \times \mathbf{U}(1)^{n-2}$; for example, an eigenvector basis for hyperisospin

$\mathbf{U}(2)$ is given by $\{e^1, e^2\}$ with

$$e^{i\alpha+\frac{\mathbf{1}_2+\sigma^3}{2}}e^{i\alpha-\frac{\mathbf{1}_2-\sigma^3}{2}}e^1 = e^{i\alpha+\frac{\mathbf{1}_2+\sigma^3}{2}}e^1$$
$$e^{i\alpha+\frac{\mathbf{1}_2+\sigma^3}{2}}e^{i\alpha-\frac{\mathbf{1}_2-\sigma^3}{2}}e^2 = e^{i\alpha-\frac{\mathbf{1}_2-\sigma^3}{2}}e^2$$

4.2. An Eigenvector Basis for the Hydrogen Atom

In the defining quartet $[\frac{1}{2};\frac{1}{2}]$ representation of the $\mathbf{SO}(4)$-invariant bound state dynamics of the hydrogen atom, the eigenvectors of the Lie algebra as basis of the representation space $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$e^1\otimes e^1 = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix},\qquad e^1\otimes e^2 = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix},\qquad e^2\otimes e^1, e^2\otimes e^2$$

have the eigenvalues, given in the following table:

	$\sigma^3\otimes\mathbf{1}_2$	$\mathbf{1}_2\otimes\tau^3$	$\mathcal{L}^3_+$	$\mathcal{L}^3_-$
$e^1\otimes e^1$	$+1$	$+1$	$+1$	0
$e^1\otimes e^2$	$+1$	-1	0	1
$e^2\otimes e^1$	-1	$+1$	0	-1
$e^2\otimes e^2$	-1	-1	-1	0

$\mathcal{L}^3_\pm = \frac{\sigma^3\otimes\mathbf{1}_2\pm\mathbf{1}_2\otimes\tau^3}{2}$

They remain eigenvectors of the correlated group $\mathbf{SO}(4)$ where they have to be characterized by the orthogonal basis $\mathcal{L}^3_+\mathcal{L}^3_- = 0$. The third angular momentum $\mathcal{L}^3_+ = \mathcal{L}^3$ component generates an axial rotation group $e^{i\gamma+\mathcal{L}^3_+} \in \mathbf{SO}(2) \subset \mathbf{SO}(3)$. The quartet comes as a $\mathbf{SO}(3)$-triplet $(e^1\otimes e^1, \frac{e^1\otimes e^2+e^2\otimes e^1}{\sqrt{2}}, e^2\otimes e^2)$ with $L = 1$ and as a singlet $\frac{e^1\otimes e^2-e^2\otimes e^1}{\sqrt{2}}$ with $L = 0$. The second basis element $\mathcal{L}^3_- = \frac{\mathcal{F}^3}{\sqrt{-2H}}$, also generates an Abelian subgroup $\mathbf{SO}(2)$, which, however, is not a subgroup of another $\mathbf{SO}(3)$. Its eigenvalue is related to the number of radial knots N in the Schrödinger wave functions $2J + 1 = L + 1 + N$ (not directly $N = L^3_-$!).

4.3. No Eigenvector Basis for Hypercharge–Isospin–Color

The defining representation of the Lie algebra $\log[\mathbf{U}(1) \times \mathbf{SU}(2) \times \mathbf{SU}(3)]$ on a complex six-dimensional space $\mathbb{C}^2 \otimes \mathbb{C}^3$ has an eigenvector basis with the eigenvalues read from the diagonal Pauli matrices given in the following table:

	$Y\mathbf{1}_2 \otimes \mathbf{1}_3$	$\tau^3 \otimes \mathbf{1}_3$	$\mathbf{1}_2 \otimes \sqrt{3}\lambda^8$	$\mathbf{1}_2 \otimes \lambda^3$
$e^1 \otimes e^1$	Y	$+1$	$+1$	$+1$
$e^1 \otimes e^2$	Y	$+1$	$+1$	-1
$e^1 \otimes e^3$	Y	$+1$	-2	0
$e^2 \otimes e^1$	Y	-1	$+1$	$+1$
$e^2 \otimes e^2$	Y	-1	$+1$	-1
$e^2 \otimes e^3$	Y	-1	-2	0

The normalization $Y \in \mathbb{R}$ will be discussed hereafter.

Without a complete Cartan torus there do not exist eigenvector bases for the correlated group $\mathbf{U}(2 \times 3)$ in faithful representations.

The subset of those $\mathbf{U}(2 \times 3)$-representations which are trivial either for color or for isospin, that is, the representations of hyperisospin $\mathbf{U}(2)$ or hypercolor $\mathbf{U}(3)$, allow eigenvector bases for $\mathbf{U}(2)$ and $\mathbf{U}(3)$, respectively. They are obtained from the corresponding fundamental representations given by the antisymmetric cube or the antisymmetric square of the defining $\mathbf{U}(2 \times 3)$-representation which triples and doubles the hypercharge normalization. Those product representations have the eigenvector bases

$\bigwedge^3 u \in \mathbf{U}(2)$ on $\mathbb{C}^2$ with $3Y = 1$

	$3Y\mathbf{1}_2$	$\mathbf{w}_2$	$\mathcal{P}(2)_+$	$\mathcal{P}(2)_-$
e^1	$3Y$	$+1$	1	0
e^2	$3Y$	-1	0	1

$\mathbf{w}_2 = \tau^3;\ \mathcal{P}(2)_\pm = \frac{\mathbf{1}_2 \pm \mathbf{w}_2}{2}$

$\bigwedge^2 u \in \mathbf{U}(3)$ on $\mathbb{C}^3$ with $2Y = 1$

	$2Y\mathbf{1}_3$	$\mathbf{w}_3$	λ^3	$\mathcal{P}(3)_+$	$\mathcal{P}(3)_-$
e^1	$2Y$	$+1$	$+1$	1	0
e^2	$2Y$	$+1$	-1	1	0
e^3	$2Y$	-2	0	0	1

$\mathbf{w}_3 = \sqrt{3}\lambda^8;\ \mathcal{P}(3)_+ = \frac{2\mathbf{1}_3 + \mathbf{w}_3}{3}; \mathcal{P}(3)_- = \frac{\mathbf{1}_3 - \mathbf{w}_3}{3}$

To obtain the projectors, the normalization Y has to fullfill $3|Y| = 1$ for $\mathbf{U}(2)$ and $2|Y| = 1$ for $\mathbf{U}(3)$.

It is impossible to give an eigenvector basis for the internal group $\mathbf{U}(2 \times 3)$ in faithful representations, for example, for the left-handed isodoublet color triplet quark representation $[\frac{1}{6}||1; 1, 0]$. It is possible to give eigenvector bases for the reduced internal groups $\mathbf{U}(2)$ or $\mathbf{U}(3)$, for example, for the representations with the left-handed isodoublet color singlet lepton or the right-handed isosinglet color triplet quarks, respectively. A quark confinement can be interpreted as the decision with respect to a particle classification for the complete Cartan torus $\mathbf{U}(1)_+ \times \mathbf{U}(1)_- \subset \mathbf{U}(2)$ for hyperisospin and against the complete Cartan torus $\mathbf{U}(1)_+ \times \mathbf{U}(1)_- \times \mathbf{U}(1)_3 \subset \mathbf{U}(3)$ for hypercolor.

With the reduction from $\mathbf{U}(2 \times 3)$ to hyperisospin $\mathbf{U}(2)$, the projector basis $\frac{\mathbf{1}_2 \pm \tau^3}{2}$ generates the electromagnetic subgroup $\mathbf{U}(1)$ as one factor in the Cartan torus, say $\mathbf{U}(1)_+$. With the choice of a projector basis to characterize eigenstates, no reduction from the interaction hyperisospin symmetry $\mathbf{U}(2)$ to the particle electromagnetic symmetry $\mathbf{U}(1)_+$ is enforced.

5. THE EXTERNAL–INTERNAL SYMMETRY CORRELATION

Also the external Lorentz group and the internal hyperisospin-color group for the interaction symmetry transformations are centrally correlated.

5.1. Correlations by Defining Representations

Correlations are implementable by specific representations, especially by defining representations.

A rank r semisimple Lie algebra, for example, $\log \mathbf{SU}(n)$ with $r = n - 1$, has r-fundamental representations, for example, quark and antiquark representations $[1, 0]$ and $[0, 1]$ for $\log \mathbf{SU}(3)$, which are a basis—with respect to totally symmetric tensor products—for all representations, for example, $[2C_1, 2C_2] \subseteq \overset{2C_1}{\vee}[1, 0] \otimes \overset{2C_2}{\vee}[0, 1]$. A layer deeper are the defining representations which are a subset of the fundamental representations and allow, also using totally antisymmetric products, to construct all fundamental representations, for example, antitriplet from triplets $[0, 1] \cong [1, 0] \wedge [1, 0]$. If such a defining representation comes with a central correlation of the represented groups, all its products will inherit this correlation.

The complex defining representation of $\mathbf{SU}(n)$ on $\mathbb{C}^n$ comes with a representation of the scalar phase $\mathbf{U}(\mathbf{1}_n)$

$$\begin{aligned} \mathbf{U}(n) \ni e^{i\alpha_0 Y \mathbf{1}_n + i\vec{\alpha}\vec{\sigma}(n)} &= [Y \| \underbrace{1, 0 \ldots, 0}_{n-1 \text{ places}}] \\ \text{e.g., } \mathbf{U}(2) \ni e^{i\alpha_0 Y \mathbf{1}_2 + i\vec{\alpha}\vec{\sigma}} &= [Y \| 1] \\ \mathbf{U}(3) \ni e^{i\alpha_0 Y \mathbf{1}_2 + i\vec{\alpha}\vec{\lambda}} &= [Y \| 1, 0] \end{aligned}$$

The correlation from the $\mathbf{U}(n) \cong \frac{\mathbf{U}(1) \times \mathbf{SU}(n)}{\mathbf{I}(n)}$ representation is inherited by all products, for example, for the antisymmetric ones with n-ality k mod n

$$\begin{aligned} \bigwedge^{k} [Y \| 1, 0, \ldots, 0] &= [kY \| \underbrace{0, \ldots, 0, 1, 0, \ldots, 0}_{k\text{th place}}], \quad k = 1, \ldots, n-1 \\ \bigwedge^{n} [Y \| 1, 0, \ldots, 0] &= [nY \| 0, \ldots, 0, 0], \quad nY \in \mathbb{Z} \end{aligned}$$

The $\mathbf{U}(1)$-representation for power n has to come with an integer winding number, minimal for $|nY| = 1$. For the examples above one obtains with minimal hypercharge Y

$$\mathbf{U}(2)\colon \begin{cases} [Y\|1] \wedge [Y\|1] = [2Y\|0] \in \mathbf{U}(1), & |Y| = \frac{1}{2} \\ \bigvee^{2T} [Y\|1] = [2TY|2T|, & 2T = 0, 1, \ldots \end{cases}$$

$$\mathbf{U}(3)\colon \begin{cases} [Y\|1, 0] \wedge [Y\|1, 0] = [2Y\|0, 1] \\ \bigwedge^{3} [Y\|1, 0] = [3Y\|0, 0] \in \mathbf{U}(1), \quad |Y| = \frac{1}{3} \end{cases}$$

In this way, if all interaction parametrizing fields of the standard model arise as representation products of one defining complex six-dimensional representation on $\mathbb{C}^2 \otimes \mathbb{C}^3$, they display the central $\mathbb{I}(6)$-correlation as given in $\frac{\mathbf{U}(1)\times\mathbf{SU}(2)\times\mathbf{SU}(3)}{\mathbf{I}(6)}$, for example, for the fermion fields

$$u = \left[\frac{1}{6}\middle\|1; 1, 0\right] \Rightarrow \begin{cases} \text{quark isodoublet } \mathbf{q} & \text{with } u = \left[\frac{1}{6}\middle\|1; 1, 0\right] \\ \text{down antiquark isosinglet } \mathbf{d}^* & \text{with } \bigwedge^{2} u = \left[\frac{1}{3}\middle\|0; 0, 1\right] \\ \text{antilepton isodoublet } \mathbf{1}^* & \text{with } \bigwedge^{3} u = \left[\frac{1}{2}\middle\|1; 0, 0\right] \\ \text{up quark isosinglet } \mathbf{u} & \text{with } \bigwedge^{4} u = \left[\frac{2}{3}\middle\|0; 1, 0\right] \\ \text{lepton isosinglet } \mathbf{e}^* & \text{with } \bigwedge^{6} u = [1|0; 0, 0] \end{cases}$$

Similarily, the defining quartet (2s and 2p states) representation $[\frac{1}{2}; \frac{1}{2}]$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ for the bound states of the hydrogen atom gives rise to all bound state representations $[J; J]$ arising as direct summands in one of the totally symmetric products $\bigvee^{N}[\frac{1}{2}; \frac{1}{2}]$ acting on $\mathbb{C}^{2\binom{3+N}{3}}$

$$\bigvee^{N} \left[\frac{1}{2}; \frac{1}{2}\right] = \begin{cases} \bigoplus_{J=0,1,\cdots,\frac{N}{2}} [J; J], & N \text{ even} \Rightarrow 2J+1 \text{ odd} \\ \bigoplus_{J=\frac{1}{2},\frac{3}{2},\cdots,\frac{N}{2}} [J; J], & N \text{ odd} \Rightarrow 2J+1 \text{ even} \end{cases}$$

All these representations inherit the $\mathbb{I}(2)$-correlation in $\frac{\mathbf{SU}(2)\times\mathbf{SU}(2)}{\mathbf{I}(2)}$, nontrivial for even multiplicities $(2J+1)^2 = 4, 16, \ldots$.

5.2. The Spin–Isospin Correlation

If the hadrons arise from quark field products they inherit the $\mathbb{I}(2)$-correlation from Lorentz $\mathbf{SL}(\mathbb{C}^2)$ and isospin $\mathbf{SU}(2)$ in the fundamental representation on $\mathbb{C}^2 \otimes \mathbb{C}^2$, as seen in the left-handed Weyl doublet isodoublet color triplet quark representation on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$, faithful for the centrally correlated group

$$\frac{\mathbf{SL}(\mathbb{C}^2) \times \mathbf{U}(1) \times \mathbf{SU}(2) \times \mathbf{SU}(3)}{\mathbf{I}(2) \times \mathbf{I}(2) \times \mathbf{I}(3)}$$

and arising also in the left-handed Weyl doublet isodoublet color singlet lepton representation on $\mathbb{C}^2 \otimes \mathbb{C}^2$, faithful for

$$\frac{\mathbf{SL}(\mathbb{C}^2) \times \mathbf{U}(1) \times \mathbf{SU}(2)}{\mathbf{I}(2) \times \mathbf{I}(2)}$$

One factor $\mathbb{I}(2)$ correlates spin in $\mathbf{SL}(\mathbb{C}^2)$ with isospin $\mathbf{SU}(2)$, the other factor $\mathbb{I}(2)$ isospin $\mathbf{SU}(2)$ with hypercharge $\mathbf{U}(1)$.

A three-dimensional Cartan subalgebra for a maximal compact seven-dimensional Lie subalgebra for spin, hypercharge, and isospin

$$\mathbb{R}^7 \cong \log[\mathbf{SU}(2) \times \mathbf{U}(1) \times \mathbf{SU}(2)] \subset \log[\mathbf{SL}(\mathbb{C}^2) \times \mathbf{U}(1) \times \mathbf{SU}(2)] \cong \mathbb{R}^{10}$$

in a fundamental complex four-dimensional representation is given by

$$\left\{ i\omega_3\sigma^3 \otimes \mathbf{1}_2 + i\alpha_0\mathbf{1}_2 \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes i\alpha_3\tau^3 \right\} \cong \mathbb{R}^3$$

The exponent involves four parameters

$$\mathbf{U}(1)_{\sigma^3} \circ \mathbf{U}(\mathbf{1}_2) \circ \mathbf{U}(1)_{\tau^3} = \mathbf{U}(1)_{\sigma^3} \circ [\mathbf{U}(1)_+ \times \mathbf{U}(1)_-]$$
$$\text{parameters: } \{\pm\omega_3 + \alpha_0 \pm \alpha_3\} = \{\pm\omega_3 + \alpha_+, \pm\omega_3 + \alpha_-\}$$

which prevent a complete three-dimensional Cartan torus for $\frac{\mathbf{SU}(2) \times \mathbf{U}(1) \times \mathbf{SU}(2)}{\mathbf{I}(2) \times \mathbf{I}(2)}$. There exist complete two-dimensional Cartan tori $\mathbf{U}(1) \times \mathbf{U}(1)$ for the centrally correlated two factor subgroups

$$\frac{\mathbf{SU}(2) \times \mathbf{U}(1)}{\mathbf{I}(2)} \cong \mathbf{U}(2), \qquad \frac{\mathbf{SU}(2) \times \mathbf{SU}(2)}{\mathbf{I}(2)} \cong \mathbf{SO}(4)$$

Therefore, one has to decide with respect to eigenvector bases once more for a subgroup with a two-dimensional Cartan torus—the choice in the observed particles is $\mathbf{U}(2)$ with the scalar phase factor the electromagnetic $\mathbf{U}(1)_+ \subset \mathbf{U}(2)$ from hyperisospin

$$\mathbf{U}(2) \cong \frac{\mathbf{SU}(2) \times \mathbf{U}(1)_+}{\mathbf{I}(2)} \supset \mathbf{U}(1)_{\sigma^3} \circ \mathbf{U}(1)_+, \qquad \text{parameters: } \{\pm\omega_3 + \alpha_+\}$$

$$\text{Cartan torus: } e^{i(\omega_3+\alpha_+)\frac{\mathbf{1}_2+\sigma^3}{2} \otimes \frac{\mathbf{1}_2+\tau^3}{2}} e^{i(-\omega_3+\alpha_+)\frac{\mathbf{1}_2-\sigma^3}{2} \otimes \frac{\mathbf{1}_2+\tau^3}{2}} \in \mathbf{U}(1)_{++} \times \mathbf{U}(1)_{-+}$$

The other hyperisospin circle $\mathbf{U}(1)_-$ does not arise with eigenvectors. In the standard model the $\mathbf{U}(1)_-$ symmetry is spontaneously broken via a degenerated ground state, implemented by the Higgs field Φ in a defining $\mathbf{U}(2)$-representation

$$\langle \Phi \otimes \Phi^* \rangle = \begin{pmatrix} 0 \\ M \end{pmatrix} \otimes (0, M) = \begin{pmatrix} 0 & 0 \\ 0 & M^2 \end{pmatrix} = \frac{\mathbf{1}_2 - \tau^3}{2} M^2$$

The group $\mathbf{U}(2)$ induces nontrivial isospin multiplicities in the representation space (particles as translation eigenvectors) in contrast to the confined color.

6. SUMMARY

The construction of eigenstates for the large homogeneous interaction symmetry group can be done in three steps ($\downarrow$), the first two ones characterized by the choice of a maximal, but not complete Cartan torus

	Group	Defining field	Representation
Interaction operations	$\frac{\mathbf{SL}(\mathbb{C}^2) \times \mathbf{U}(1) \times \mathbf{SU}(2) \times \mathbf{SU}(3)}{\mathbf{I}(2) \times \mathbf{I}(6)}$	$\psi_{\alpha,i}$, $\alpha = 1, 2$, $i = 1, 2, 3$	With $\left[\frac{1}{6} \Vert 1; 1, 0\right]$
Confinement of color $\mathbf{SU}(3)$	$\downarrow$		
	$\frac{\mathbf{SL}(\mathbb{C}^2) \times \mathbf{U}(1) \times \mathbf{SU}(2)}{\mathbf{I}(2) \times \mathbf{I}(2)}$	$(\overset{3}{\bigwedge} \psi)\alpha$, $\alpha = 1, 2$	With $\left[\frac{1}{2} \Vert\ 1\right]$
Reduction $\mathbf{U}(2) \to \mathbf{U}(1)_+$ to charge	$\downarrow$		
	$\frac{\mathbf{SL}(\mathbb{C}^2) \times \mathbf{U}(1)_+}{\mathbf{I}(2)}$	$\mathbf{p}_+ = \Phi^\alpha (\overset{3}{\bigwedge} \psi)\alpha$, $\mathbf{n}_0 = \Phi^*_\beta \epsilon^{\beta\alpha} (\overset{3}{\bigwedge} \psi)\alpha$	With [1], [0]
Rest or momentum system	$\downarrow$		
Particles	$m^2 > 0$: $\frac{\mathbf{SU}(2) \times \mathbf{U}(1)_+}{\mathbf{I}(2)} \cong \mathbf{U}(2)$; $m = 0$: $\frac{\mathbf{U}(1)_3 \times \mathbf{U}(1)_+}{\mathbf{I}(2)} \cong \mathbf{U}(1) \times \mathbf{U}(1)$		

In the third and fourth column only the internal representation properties are given.

REFERENCES

Baird, G. E. and Biedenharn, L. C. (1964). In *Proceedings 1st Goral Gabels Conference on Symmetry Principles*, Freeman, p. 68.
Boerner, H. (1955). *Darstellungen von Gruppen*, Springer, Berlin, Göttingen, Heidelberg.
Bourbaki, N. (1989). *Lie Groups and Lie Algebras*, chaps. 1–3, Springer, Berlin, Heidelberg, New York, London, Paris, Tokyo.
Fock, V. (1935). *Zeitschrift für Physik* **98**, 145.
Fulton, W. and Harris, J. (1991). *Representation Theory*, Springer.
Helgason, S. (1978). *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York.
Hucks, J. (1991). *Physical Review D* **43**, 2709.
Mackey, G. W. (1968). *Induced Representations of Group and Qauntum Mechanics*, W. A. Benjamin, New York, Amsterdam.
O'Raifeartaigh, L. (1986). *Group Structure of Gauge Theories*, Cambridge University Press, Cambridge.
Weinberg, S. (1967). *Physical Review Letters* **18**, 507.
Wigner, E. P. (1939). *Annals of Mathematics* **40**, 149.
Saller, H. (1991). *Nuovo Cimento* **104A**, 203.
Saller, H. (1992). *Nuovo Cimento* **105A**, 1745.
Saller, H. (1993). *Nuovo Cimento* **105A**, 1189.
Saller, H. (1994). *Nuovo Cimento* **107A**, 1679.
Saller, H. (1998). *Nuovo Cimento* **111A**, 1375.
Saller, H. (1998). *International Journal of Theoretical Physics* **37**, 2333.